

Strong instability of standing waves for nonlinear Schrödinger equations with a delta potential

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Abstract

We study strong instability (instability by blowup) of standing wave solutions for a nonlinear Schrödinger equation with an attractive delta potential and L^2 -supercritical power nonlinearity in one space dimension. We also compare our sufficient condition on strong instability with some known results on orbital instability.

1 Introduction

In our previous paper [18], we studied the strong instability (instability by blowup) of standing wave solutions $e^{i\omega t}\phi_\omega(x)$ for the following nonlinear Schrödinger equation with double power nonlinearity:

$$i\partial_t u = -\Delta u - a|u|^{p-1}u - b|u|^{q-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \quad (1.1)$$

where a and b are positive constants, $1 < p < 1 + 4/N < q < 2^* - 1$. Here, 2^* is defined by $2^* = 2N/(N-2)$ if $N \geq 3$, and $2^* = \infty$ if $N = 1, 2$. For ground states ϕ_ω , we proved that the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is strongly unstable for sufficiently large ω . Moreover, we announced in [18]

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that our method of proof is not restricted to the double power case (1.1), but is also applicable to other type of nonlinear Schrödinger equations.

In this paper, we consider the following nonlinear Schrödinger equation with a delta potential in one space dimension:

$$i\partial_t u = -\partial_x^2 u - \gamma\delta(x)u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad (1.2)$$

where $\gamma \in \mathbb{R}$ is a constant, $\delta(x)$ is the delta measure at the origin, and $1 < p < \infty$. The equations of the form (1.2) arise in a wide variety of physical models with a point defect on the line, and have been studied by many authors (see, e.g., [4, 5, 7, 10, 11, 12, 13, 15] and references therein).

We study the strong instability of standing wave solutions $e^{i\omega t}\phi_\omega(x)$ of (1.2), where $\omega > \gamma^2/4$, and

$$\phi_\omega(x) = \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2}|x| + \tanh^{-1} \left(\frac{\gamma}{2\sqrt{\omega}} \right) \right) \right\}^{\frac{1}{p-1}}, \quad (1.3)$$

which is a unique positive solution of

$$-\partial_x^2 \phi + \omega \phi - \gamma\delta(x)\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}. \quad (1.4)$$

The well-posedness of the Cauchy problem for (1.2) in the energy space $H^1(\mathbb{R})$ follows from an abstract result in Cazenave [2] (see Theorem 3.7.1 and Corollary 3.3.11 in [2], and also Section 2 of [5]).

Proposition 1.1. *For any $u_0 \in H^1(\mathbb{R})$ there exist $T_{\max} = T_{\max}(u_0) \in (0, \infty]$ and a unique solution $u \in C([0, T_{\max}), H^1(\mathbb{R}))$ with $u(0) = u_0$ such that either $T_{\max} = \infty$ (global existence) or $T_{\max} < \infty$ and $\lim_{t \rightarrow T_{\max}} \|\partial_x u(t)\|_{L^2} = \infty$ (finite time blowup). Furthermore, the solution $u(t)$ satisfies*

$$E(u(t)) = E(u_0), \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 \quad (1.5)$$

for all $t \in [0, T_{\max})$, where the energy E is defined by

$$E(v) = \frac{1}{2} \|\partial_x v\|_{L^2}^2 - \frac{\gamma}{2} |v(0)|^2 - \frac{1}{p+1} \|v\|_{L^{p+1}}^{p+1}.$$

Here, we give the definitions of stability and instability of standing waves.

Definition 1.2. We say that the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.2) is *orbitally stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|u_0 - \phi_\omega\|_{H^1} < \delta$, then the solution $u(t)$ of (1.2) with $u(0) = u_0$ exists globally and satisfies

$$\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta} \phi_\omega\|_{H^1} < \varepsilon.$$

Otherwise, $e^{i\omega t}\phi_\omega$ is said to be *orbitally unstable*.

Definition 1.3. We say that $e^{i\omega t}\phi_\omega$ is *strongly unstable* if for any $\varepsilon > 0$ there exists $u_0 \in H^1(\mathbb{R})$ such that $\|u_0 - \phi_\omega\|_{H^1} < \varepsilon$ and the solution $u(t)$ of (1.2) with $u(0) = u_0$ blows up in finite time.

Before we state our main result, we recall some known results. First, we consider the nonlinear Schrödinger equation without potential:

$$i\partial_t u = -\partial_x^2 u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}. \quad (1.6)$$

Let $1 < p < \infty$, $\omega > 0$ and

$$\varphi_\omega(x) = \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2} x \right) \right\}^{\frac{1}{p-1}}.$$

When $1 < p < 5$, the standing wave solution $e^{i\omega t}\varphi_\omega$ of (1.6) is orbitally stable for all $\omega > 0$ (see [3]). When $p \geq 5$, $e^{i\omega t}\varphi_\omega$ is strongly unstable for all $\omega > 0$ (see [1] and also [2]).

Next, we consider the attractive potential case $\gamma > 0$ in (1.2), which was first studied by Goodman, Holmes and Weinstein [7] for the case $p = 3$, and then by Fukuizumi, Ohta and Ozawa [5] for $1 < p < \infty$. The following is proved in [5].

Proposition 1.4 ([5]). *Let $\gamma > 0$ and $\omega > \gamma^2/4$.*

- (i) *When $1 < p \leq 5$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.2) is orbitally stable for any $\omega \in (\gamma^2/4, \infty)$.*
- (ii) *When $p > 5$, there exists $\omega_0 = \omega_0(p, \gamma) \in (\gamma^2/4, \infty)$ such that the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.2) is orbitally stable for any $\omega \in (\gamma^2/4, \omega_0)$, and it is orbitally unstable for any $\omega \in (\omega_0, \infty)$. Here, $\omega_0(p, \gamma) = \gamma^2/[4\xi_0(p)^2]$ and $\xi_0(p) \in (0, 1)$ is a unique solution of*

$$\frac{p-5}{p-1} \int_{\xi}^1 (1-s^2)^{\frac{2}{p-1}-1} ds = \xi (1-\xi^2)^{\frac{2}{p-1}-1} \quad (0 < \xi < 1). \quad (1.7)$$

Remark 1. To prove Proposition 1.4, the following sufficient conditions for orbital stability and instability are used (see [8, 9, 19, 20, 21]).

Let $p > 1$, $\gamma > 0$ and $\omega > \gamma^2/4$.

- (i) If $\partial_\omega \|\phi_\omega\|_{L^2}^2 > 0$ at $\omega = \hat{\omega}$, then $e^{i\hat{\omega}t}\phi_{\hat{\omega}}$ is orbitally stable.
- (ii) If $\partial_\omega \|\phi_\omega\|_{L^2}^2 < 0$ at $\omega = \hat{\omega}$, then $e^{i\hat{\omega}t}\phi_{\hat{\omega}}$ is orbitally unstable.

By the formula (1.3), we have

$$\begin{aligned}\|\phi_\omega\|_{L^2}^2 &= 2 \int_0^\infty \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2} x + \tanh^{-1} \xi(\omega, \gamma) \right) \right\}^{\frac{2}{p-1}} dx \\ &= \frac{4}{(p-1)\sqrt{\omega}} \left(\frac{(p+1)\omega}{2} \right)^{\frac{2}{p-1}} \int_{\tanh^{-1} \xi(\omega, \gamma)}^\infty (\operatorname{sech}^2 y)^{\frac{2}{p-1}} dy,\end{aligned}$$

where we put

$$\xi(\omega, \gamma) = \frac{\gamma}{2\sqrt{\omega}}. \quad (1.8)$$

Moreover, for $0 < a < 1$ and $\beta > 0$, we have

$$\int_{\tanh^{-1} a}^\infty (\operatorname{sech}^2 y)^\beta dy = \int_a^1 (1-s^2)^{\beta-1} ds. \quad (1.9)$$

Thus, we obtain

$$\begin{aligned}\|\phi_\omega\|_{L^2}^2 &= \frac{4}{p-1} \left(\frac{p+1}{2} \right)^{\frac{2}{p-1}} \left(\frac{2}{\gamma} \right)^{\frac{p-5}{p-1}} F(\xi(\omega, \gamma)), \\ F(\xi) &= \xi^{\frac{p-5}{p-1}} \int_\xi^1 (1-s^2)^{\frac{2}{p-1}-1} ds.\end{aligned}$$

Then, since $\partial_\omega \xi(\omega, \gamma) < 0$, for $\xi = \xi(\omega, \gamma)$, we see that

$$\begin{aligned}\partial_\omega \|\phi_\omega\|_{L^2}^2 < 0 &\iff F'(\xi) > 0 \\ &\iff \frac{p-5}{p-1} \int_\xi^1 (1-s^2)^{\frac{2}{p-1}-1} ds > \xi (1-\xi^2)^{\frac{2}{p-1}-1} \\ &\iff \xi < \xi_0(p) \iff \omega > \omega_0(p, \gamma).\end{aligned}$$

Remark 2. For the borderline case $\omega = \omega_0$ in Proposition 1.4 (ii), the standing wave solution $e^{i\omega_0 t} \phi_{\omega_0}$ of (1.2) is orbitally unstable (see [17]).

Now we state our main result in this paper.

Theorem 1.5. *Let $\gamma > 0$, $p > 5$, $\omega > \gamma^2/4$, and let ϕ_ω be the function defined by (1.3). Let $\xi_1(p) \in (0, 1)$ be a unique solution of*

$$\frac{p-5}{p-1} \int_\xi^1 (1-s^2)^{\frac{2}{p-1}} ds = \xi (1-\xi^2)^{\frac{2}{p-1}} \quad (0 < \xi < 1), \quad (1.10)$$

and define $\omega_1 = \omega_1(p, \gamma) = \gamma^2/[4\xi_1(p)^2]$. Then, the standing wave solution $e^{i\omega t} \phi_\omega$ of (1.2) is strongly unstable for all $\omega \in (\omega_1, \infty)$.

Remark 3. The condition $\omega > \omega_1$ in Theorem 1.5 is equivalent to $E(\phi_\omega) > 0$ (see Theorem 1.6 below).

Remark 4. For the repulsive potential case $\gamma < 0$, it is proved in [15] that if $p \geq 5$, the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.2) is strongly unstable for all $\omega \in (\gamma^2/4, \infty)$. The situation for the attractive potential case $\gamma > 0$ is quite different from the case $\gamma < 0$, and we need a new approach to prove Theorem 1.5.

For $\gamma > 0$, $p > 1$ and $\omega > \gamma^2/4$, we define functionals S_ω and K_ω on $H^1(\mathbb{R})$ by

$$\begin{aligned} S_\omega(v) &= \frac{1}{2}\|\partial_x v\|_{L^2}^2 + \frac{\omega}{2}\|v\|_{L^2}^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|_{L^{p+1}}^{p+1}, \\ K_\omega(v) &= \|\partial_x v\|_{L^2}^2 + \omega\|v\|_{L^2}^2 - \gamma|v(0)|^2 - \|v\|_{L^{p+1}}^{p+1}. \end{aligned}$$

Note that (1.4) is equivalent to $S'_\omega(\phi) = 0$, and

$$K_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1} = \langle S'_\omega(v), v \rangle$$

is the so-called Nehari functional.

We denote the set of nontrivial solutions of (1.4) by

$$\mathcal{A}_\omega = \{v \in H^1(\mathbb{R}^N) : S'_\omega(v) = 0, v \neq 0\},$$

and define the set of ground states of (1.4) by

$$\mathcal{G}_\omega = \{\phi \in \mathcal{A}_\omega : S_\omega(\phi) \leq S_\omega(v) \text{ for all } v \in \mathcal{A}_\omega\}. \quad (1.11)$$

Moreover, consider the minimization problem:

$$d(\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}^N), K_\omega(v) = 0, v \neq 0\}. \quad (1.12)$$

Then, for any $\omega > \gamma^2/4$, we have

$$\begin{aligned} \mathcal{A}_\omega &= \mathcal{G}_\omega = \{\phi \in H^1(\mathbb{R}^N) : S_\omega(\phi) = d(\omega), K_\omega(\phi) = 0\} \\ &= \{e^{i\theta}\phi_\omega : \theta \in \mathbb{R}\}, \end{aligned}$$

where ϕ_ω is the function defined by (1.3) (see [5, 15]).

On the other hand, the proof of finite time blowup for (1.2) relies on the virial identity. If $u_0 \in \Sigma := \{v \in H^1(\mathbb{R}) : |x|v \in L^2(\mathbb{R})\}$, then the solution $u(t)$ of (1.2) with $u(0) = u_0$ belongs to $C([0, T_{\max}), \Sigma)$, and satisfies

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8P(u(t)) \quad (1.13)$$

for all $t \in [0, T_{\max})$, where

$$P(v) = \|\partial_x v\|_{L^2}^2 - \frac{\gamma}{2}|v(0)|^2 - \frac{\alpha}{p+1}\|v\|_{L^{p+1}}^{p+1}, \quad \alpha := \frac{p-1}{2}.$$

For the proof of the virial identity (1.13), see Proposition 6 in [15].

Note that for the scaling $v^\lambda(x) = \lambda^{1/2}v(\lambda x)$ for $\lambda > 0$, we have

$$\begin{aligned} \|\partial_x v^\lambda\|_{L^2}^2 &= \lambda^2 \|\partial_x v\|_{L^2}^2, \quad |v^\lambda(0)|^2 = \lambda |v(0)|^2, \quad \|v^\lambda\|_{L^{p+1}}^{p+1} = \lambda^\alpha \|v\|_{L^{p+1}}^{p+1}, \\ \|v^\lambda\|_{L^2}^2 &= \|v\|_{L^2}^2, \quad P(v) = \partial_\lambda E(v^\lambda)|_{\lambda=1}. \end{aligned}$$

The method of Berestycki and Cazenave [1] for the nonlinear Schrödinger equations without potential (1.6) is based on the fact that $d(\omega) = S_\omega(\phi_\omega)$ can be characterized as

$$d(\omega) = \inf\{S_\omega(v) : v \in H^1(\mathbb{R}), P(v) = 0, v \neq 0\} \quad (1.14)$$

for the case $p \geq 5$. Using this fact, it is proved in [1] that if $u_0 \in \Sigma \cap \mathcal{B}_\omega^{BC}$, then the solution $u(t)$ of (1.6) with $u(0) = u_0$ blows up in finite time, where

$$\mathcal{B}_\omega^{BC} = \{v \in H^1(\mathbb{R}) : S_\omega(v) < d(\omega), P(v) < 0\}.$$

We remark that (1.14) does not hold for (1.2) with $\gamma > 0$.

On the other hand, Zhang [22] and Le Coz [14] gave an alternative proof of the result of Berestycki and Cazenave [1] for (1.6). Instead of solving the minimization problem (1.14), they [22, 14] proved that

$$d(\omega) \leq \inf\{S_\omega(v) : v \in H^1(\mathbb{R}), P(v) = 0, K_\omega(v) < 0\} \quad (1.15)$$

holds for all $\omega > 0$ if $p \geq 5$. Using this fact, it is proved in [22, 14] that if $u_0 \in \Sigma \cap \mathcal{B}_\omega^{ZL}$, then the solution $u(t)$ of (1.6) with $u(0) = u_0$ blows up in finite time, where

$$\mathcal{B}_\omega^{ZL} = \{v \in H^1(\mathbb{R}) : S_\omega(v) < d(\omega), P(v) < 0, K_\omega(v) < 0\}.$$

Note that this method can be applied to (1.2) for the repulsive potential case $\gamma < 0$ (see [15]), but not for the attractive potential case $\gamma > 0$.

In this paper, we use and modify the idea of Zhang [22] and Le Coz [14] to prove Theorem 1.5. For $\omega \in (\gamma^2/4, \infty)$ with $E(\phi_\omega) > 0$, we introduce a new set

$$\begin{aligned} \mathcal{B}_\omega &= \{v \in H^1(\mathbb{R}) : 0 < E(v) < E(\phi_\omega), \|v\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2, \\ &\quad P(v) < 0, K_\omega(v) < 0\}. \end{aligned} \quad (1.16)$$

Then, we have the following.

Theorem 1.6. *Let $\gamma > 0$, $p > 5$, $\omega > \gamma^2/4$, and assume that ϕ_ω satisfies $E(\phi_\omega) > 0$. If $u_0 \in \Sigma \cap \mathcal{B}_\omega$, then the solution $u(t)$ of (1.2) with $u(0) = u_0$ blows up in finite time.*

The rest of the paper is organized as follows. In Section 2, using the same method as in our previous paper [18], we prove Theorem 1.6. In Section 3, we show that the sufficient condition $E(\phi_\omega) > 0$ in Theorem 1.6 holds if and only if $\omega > \omega_1$, and prove Theorem 1.5 using Theorem 1.6. Finally, in Section 4, we compare our sufficient condition for strong instability with some known results for orbital instability.

2 Proof of Theorem 1.6

In this section, we prove Theorem 1.6. As we have already noticed in §1, the proof of Theorem 1.6 for (1.2) is almost the same as that for (1.1) given in [18]. For the sake of completeness, we repeat the argument in [18].

Throughout this section, we assume that

$$\gamma > 0, \quad p > 5, \quad \omega > \frac{\gamma^2}{4}, \quad E(\phi_\omega) > 0.$$

Recall that $\alpha := \frac{p-1}{2} > 2$, and for the scaling $v^\lambda(x) = \lambda^{1/2}v(\lambda x)$, we have

$$E(v^\lambda) = \frac{\lambda^2}{2} \|\partial_x v\|_{L^2}^2 - \frac{\gamma\lambda}{2} |v(0)|^2 - \frac{\lambda^\alpha}{p+1} \|v\|_{L^{p+1}}^{p+1}, \quad (2.1)$$

$$P(v^\lambda) = \lambda^2 \|\partial_x v\|_{L^2}^2 - \frac{\gamma\lambda}{2} |v(0)|^2 - \frac{\alpha\lambda^\alpha}{p+1} \|v\|_{L^{p+1}}^{p+1} = \lambda \partial_\lambda E(v^\lambda), \quad (2.2)$$

$$K_\omega(v^\lambda) = \lambda^2 \|\partial_x v\|_{L^2}^2 + \omega \|v\|_{L^2}^2 - \gamma\lambda |v(0)|^2 - \lambda^\alpha \|v\|_{L^{p+1}}^{p+1}. \quad (2.3)$$

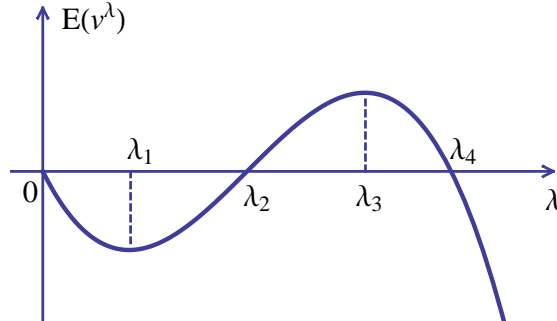


Figure 1. The graph of $\lambda \mapsto E(v^\lambda)$ for the case $E(v) > 0$.

Lemma 2.1. *If $v \in H^1(\mathbb{R})$ satisfies $E(v) > 0$, then there exist $\lambda_k = \lambda_k(v)$ ($k = 1, 2, 3, 4$) such that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ and*

- $E(v^\lambda)$ is decreasing in $(0, \lambda_1) \cup (\lambda_3, \infty)$, and increasing in (λ_1, λ_3) .
- $E(v^\lambda)$ is negative in $(0, \lambda_2) \cup (\lambda_4, \infty)$, and positive in (λ_2, λ_4) .
- $E(v^\lambda) < E(v^{\lambda_3})$ for all $\lambda \in (0, \lambda_3) \cup (\lambda_3, \infty)$.

Proof. Since $\gamma > 0$, $\alpha > 2$ and $E(v) > 0$, the conclusion is easily verified by drawing the graph of (2.1) (see Figure 1). \square

Lemma 2.2. *If $v \in H^1(\mathbb{R})$ satisfies $E(v) > 0$, $K_\omega(v) < 0$ and $P(v) = 0$, then $d(\omega) < S_\omega(v)$.*

Proof. We consider two functions $f(\lambda) = K_\omega(v^\lambda)$ and $g(\lambda) = E(v^\lambda)$.

Since $f(0) = \omega \|v\|_{L^2}^2 > 0$ and $f(1) = K_\omega(v) < 0$, there exists $\lambda_0 \in (0, 1)$ such that $K_\omega(v^{\lambda_0}) = 0$. Moreover, since $v^{\lambda_0} \neq 0$, it follows from (1.12) that $d(\omega) \leq S_\omega(v^{\lambda_0})$.

On the other hand, since $g'(1) = P(v) = 0$ and $g(1) = E(v) > 0$, it follows from Lemma 2.1 that $\lambda_3 = 1$ and $g(\lambda) < g(1)$ for all $\lambda \in (0, 1)$.

Thus, we have $E(v^{\lambda_0}) < E(v)$, and

$$d(\omega) \leq S_\omega(v^{\lambda_0}) = E(v^{\lambda_0}) + \frac{\omega}{2} \|v^{\lambda_0}\|_{L^2}^2 < E(v) + \frac{\omega}{2} \|v\|_{L^2}^2 = S_\omega(v).$$

This completes the proof. \square

Lemma 2.3. *The set*

$$\mathcal{B}_\omega = \{v \in H^1(\mathbb{R}) : 0 < E(v) < E(\phi_\omega), \|v\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2, P(v) < 0, K_\omega(v) < 0\}$$

is invariant under the flow of (1.2). That is, if $u_0 \in \mathcal{B}_\omega$, then the solution $u(t)$ of (1.2) with $u(0) = u_0$ satisfies $u(t) \in \mathcal{B}_\omega$ for all $t \in [0, T_{\max})$.

Proof. Let $u_0 \in \mathcal{B}_\omega$ and let $u(t)$ be the solution of (1.2) with $u(0) = u_0$. Then, by the conservation laws (1.5), we have

$$0 < E(u(t)) = E(u_0) < E(\phi_\omega), \quad \|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2$$

for all $t \in [0, T_{\max})$.

Next, we prove that $K_\omega(u(t)) < 0$ for all $t \in [0, T_{\max})$. Suppose that this were not true. Then, since $K_\omega(u_0) < 0$ and $t \mapsto K_\omega(u(t))$ is continuous on

$[0, T_{\max})$, there exists $t_1 \in (0, T_{\max})$ such that $K_\omega(u(t_1)) = 0$. Moreover, since $u(t_1) \neq 0$, by (1.12), we have $d(\omega) \leq S_\omega(u(t_1))$. Thus, we have

$$d(\omega) \leq S_\omega(u(t_1)) = E(u_0) + \frac{\omega}{2} \|u_0\|_{L^2}^2 < E(\phi_\omega) + \frac{\omega}{2} \|\phi_\omega\|_{L^2}^2 = d(\omega).$$

This is a contradiction. Therefore, $K_\omega(u(t)) < 0$ for all $t \in [0, T_{\max})$.

Finally, we prove that $P(u(t)) < 0$ for all $t \in [0, T_{\max})$. Suppose that this were not true. Then, there exists $t_2 \in (0, T_{\max})$ such that $P(u(t_2)) = 0$. Since $E(u(t_2)) > 0$ and $K_\omega(u(t_2)) < 0$, it follows from Lemma 2.2 that $d(\omega) < S_\omega(u(t_2))$. Thus, we have

$$d(\omega) < S_\omega(u(t_2)) = E(u_0) + \frac{\omega}{2} \|u_0\|_{L^2}^2 < E(\phi_\omega) + \frac{\omega}{2} \|\phi_\omega\|_{L^2}^2 = d(\omega).$$

This is a contradiction. Therefore, $P(u(t)) < 0$ for all $t \in [0, T_{\max})$. \square

Lemma 2.4. *For any $v \in \mathcal{B}_\omega$,*

$$E(\phi_\omega) \leq E(v) - P(v).$$

Proof. Since $K_\omega(v) < 0$, as in the proof of Lemma 2.2, there exists $\lambda_0 \in (0, 1)$ such that $S_\omega(\phi_\omega) = d(\omega) \leq S_\omega(v^{\lambda_0})$. Moreover, since $\|v^{\lambda_0}\|_{L^2}^2 = \|v\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2$, we have

$$E(\phi_\omega) \leq E(v^{\lambda_0}). \quad (2.4)$$

On the other hand, since $P(v^\lambda) = \lambda \partial_\lambda E(v^\lambda)$, $P(v) < 0$ and $E(v) > 0$, it follows from Lemma 2.1 that $\lambda_3 < 1 < \lambda_4$. Moreover, since $\partial_\lambda^2 E(v^\lambda) < 0$ for $\lambda \in [\lambda_3, \infty)$, by a Taylor expansion, we have

$$E(v^{\lambda_3}) \leq E(v) + (\lambda_3 - 1)P(v) \leq E(v) - P(v). \quad (2.5)$$

Finally, by (2.4), (2.5) and the third property of Lemma 2.1, we have

$$E(\phi_\omega) \leq E(v^{\lambda_0}) \leq E(v^{\lambda_3}) \leq E(v) - P(v).$$

This completes the proof. \square

Now we give the proof of Theorem 1.6.

Proof of Theorem 1.6. Let $u_0 \in \Sigma \cap \mathcal{B}_\omega$ and let $u(t)$ be the solution of (1.2) with $u(0) = u_0$. Then, by Lemma 2.3, $u(t) \in \mathcal{B}_\omega$ for all $t \in [0, T_{\max})$.

Moreover, by Lemma 2.4 and the virial identity, we have

$$\begin{aligned} P(u(t)) &\leq E(u(t)) - E(\phi_\omega) = E(u_0) - E(\phi_\omega), \\ \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 &= 8P(u(t)) \leq 8\{E(u_0) - E(\phi_\omega)\} \end{aligned}$$

for all $t \in [0, T_{\max})$. Since $E(u_0) < E(\phi_\omega)$, this implies $T_{\max} < \infty$. This completes the proof. \square

3 Proof of Theorem 1.5

First, we prove the following lemma.

Lemma 3.1. *Let $\gamma > 0$, $p > 5$ and $\omega > \gamma^2/4$. Let $\omega_1(p, \gamma)$ be the number defined in Theorem 1.5. Then, $E(\phi_\omega) > 0$ if and only if $\omega > \omega_1(p, \gamma)$.*

Proof. Since $P(\phi_\omega) = 0$, we see that $E(\phi_\omega) > 0$ if and only if

$$\gamma |\phi_\omega(0)|^2 < \frac{p-5}{p+1} \|\phi_\omega\|_{L^{p+1}}^{p+1}. \quad (3.1)$$

Moreover, by (1.3) and (1.8), we have

$$\begin{aligned} |\phi_\omega(0)|^2 &= \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2(\tanh^{-1} \xi(\omega, \gamma)) \right\}^{\frac{2}{p-1}} \\ &= \left(\frac{(p+1)\omega}{2} \{1 - \xi(\omega, \gamma)^2\} \right)^{\frac{2}{p-1}}, \end{aligned} \quad (3.2)$$

$$\begin{aligned} \|\phi_\omega\|_{L^{p+1}}^{p+1} &= 2 \int_0^\infty \left\{ \frac{(p+1)\omega}{2} \operatorname{sech}^2 \left(\frac{(p-1)\sqrt{\omega}}{2} x + \tanh^{-1} \xi(\omega, \gamma) \right) \right\}^{\frac{p+1}{p-1}} dx \\ &= \frac{4}{(p-1)\sqrt{\omega}} \left(\frac{(p+1)\omega}{2} \right)^{\frac{p+1}{p-1}} \int_{\tanh^{-1} \xi(\omega, \gamma)}^\infty (\operatorname{sech}^2 y)^{\frac{p+1}{p-1}} dy \\ &= \frac{4}{(p-1)\sqrt{\omega}} \left(\frac{(p+1)\omega}{2} \right)^{\frac{p+1}{p-1}} \int_{\xi(\omega, \gamma)}^1 (1-s^2)^{\frac{2}{p-1}} ds, \end{aligned} \quad (3.3)$$

where we used (1.9). Thus, we see that (3.1) is equivalent to

$$\frac{p-5}{p-1} \int_{\xi(\omega, \gamma)}^1 (1-s^2)^{\frac{2}{p-1}} ds > \xi(\omega, \gamma) \{1 - \xi(\omega, \gamma)^2\}^{\frac{2}{p-1}}. \quad (3.4)$$

Moreover, by elementary computations, we see that the equation (1.10) has a unique solution $\xi_1(p)$ for each $p > 5$, and that (3.4) is equivalent to $\xi(\omega, \gamma) < \xi_1(p)$. This completes the proof. \square

Now we give the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $\omega \in (\omega_1, \infty)$. Then, by Lemma 3.1, $E(\phi_\omega) > 0$.

For $\lambda > 0$, we consider the scaling $\phi_\omega^\lambda(x) = \lambda^{1/2}\phi_\omega(\lambda x)$, and prove that there exists $\lambda_0 \in (1, \infty)$ such that $\phi_\omega^\lambda \in \mathcal{B}_\omega$ for all $\lambda \in (1, \lambda_0)$.

First, we have $\|\phi_\omega^\lambda\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2$ for all $\lambda > 0$. Next, since $P(\phi_\omega) = 0$ and $E(\phi_\omega) > 0$, by Lemma 2.1 and (2.2), there exists $\lambda_4 > 1$ such that

$$0 < E(\phi_\omega^\lambda) < E(\phi_\omega), \quad P(\phi_\omega^\lambda) < 0$$

for all $\lambda \in (1, \lambda_4)$. Finally, since $P(\phi_\omega) = 0$, we have

$$\partial_\lambda K_\omega(\phi_\omega^\lambda)|_{\lambda=1} = -\frac{(p-1)\alpha}{p+1} \|\phi_\omega\|_{L^{p+1}}^{p+1} < 0.$$

Since $K_\omega(\phi_\omega) = 0$, there exists $\lambda_0 \in (1, \lambda_4)$ such that $K_\omega(\phi_\omega^\lambda) < 0$ for all $\lambda \in (1, \lambda_0)$.

Therefore, $\phi_\omega^\lambda \in \mathcal{B}_\omega$ for all $\lambda \in (1, \lambda_0)$. Moreover, since $\phi_\omega^\lambda \in \Sigma$ for $\lambda > 0$, it follows from Theorem 1.6 that for any $\lambda \in (1, \lambda_0)$, the solution $u(t)$ of (1.2) with $u(0) = \phi_\omega^\lambda$ blows up in finite time.

Finally, since $\lim_{\lambda \rightarrow 1} \|\phi_\omega^\lambda - \phi_\omega\|_{H^1} = 0$, the proof is completed. \square

4 Final Remarks

In [6, 16], a sufficient condition on orbital instability of standing waves for some nonlinear Schrödinger equations is given. The sufficient condition by [6, 16] is different from that given by Shatah and Strauss [20], and it is applicable to (1.2).

More precisely, by [6, 16], we see that if

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} < 0, \tag{4.1}$$

then the standing wave solution $e^{i\omega t}\phi_\omega$ of (1.2) is orbitally unstable, where $\phi_\omega^\lambda(x) = \lambda^{1/2}\phi_\omega(\lambda x)$.

In this section, we compare the sufficient condition on strong instability $E(\phi_\omega) > 0$ in Theorem 1.6 with sufficient conditions on orbital instability $\partial_\omega \|\phi_\omega\|_{L^2}^2 < 0$ by [20] and (4.1) by [6, 16].

By (2.1), we have

$$\partial_\lambda^2 E(\phi_\omega^\lambda)|_{\lambda=1} = \|\partial_x \phi_\omega\|_{L^2}^2 - \frac{(p-1)(p-3)}{4(p+1)} \|\phi_\omega\|_{L^{p+1}}^{p+1}. \quad (4.2)$$

Since $P(\phi_\omega) = 0$, we see that (4.1) is equivalent to

$$\gamma |\phi_\omega(0)|^2 < \frac{(p-1)(p-5)}{2(p+1)} \|\phi_\omega\|_{L^{p+1}}^{p+1}. \quad (4.3)$$

Moreover, by (3.2) and (3.3), we see that (4.3) is equivalent to

$$\frac{p-5}{2} \int_{\xi(\omega, \gamma)}^1 (1-s^2)^{\frac{2}{p-1}} ds > \xi(\omega, \gamma) \{1 - \xi(\omega, \gamma)^2\}^{\frac{2}{p-1}}. \quad (4.4)$$

For $p > 5$, let $\xi_2(p) \in (0, 1)$ be a unique solution of

$$\frac{p-5}{2} \int_{\xi}^1 (1-s^2)^{\frac{2}{p-1}} ds = \xi (1 - \xi^2)^{\frac{2}{p-1}} \quad (0 < \xi < 1). \quad (4.5)$$

Then, we see that (4.4) is equivalent to $\xi(\omega, \gamma) < \xi_2(p)$, and that (4.1) holds if and only if $\omega > \omega_2(p, \gamma) := \gamma^2/[4\xi_2(p)^2]$.

On the other hand, as we mentioned in Remark 1, the condition $\partial_\omega \|\phi_\omega\|_{L^2}^2 < 0$ holds if and only if $\omega > \omega_0(p, \gamma)$, while as we proved in Lemma 3.1, the condition $E(\phi_\omega) > 0$ holds if and only if $\omega > \omega_1(p, \gamma)$. Recall that for $j = 0, 1, 2$,

$$\omega_j(p, \gamma) = \frac{\gamma^2}{4\xi_j(p)^2},$$

and $\xi_0(p)$, $\xi_1(p)$ and $\xi_2(p)$ are the unique solutions of (1.7), (1.10) and (4.5), respectively.

For $p > 5$, we see that $\xi_1(p) < \xi_2(p) < \xi_0(p)$, and that $\omega_0(p, \gamma) < \omega_2(p, \gamma) < \omega_1(p, \gamma)$. The graphs of the functions $\xi_0(p)$, $\xi_1(p)$ and $\xi_2(p)$ are given in Figure 2 for $5 < p \leq 10$ and in Figure 3 for $10 \leq p \leq 30$.

For $\omega \in [\omega_0(p, \gamma), \omega_1(p, \gamma)]$, the standing wave solution $e^{i\omega t} \phi_\omega$ of (1.2) is orbitally unstable, but we do not know whether it is strongly unstable or not. However, it seems natural to conjecture that $e^{i\omega t} \phi_\omega$ is strongly unstable at least for $\omega > \omega_2(p, \gamma)$. Note that some numerical results are given in Section 6.1.3 of [15] for the case where $p = 6$, $\gamma = 1$ and $\omega = 4$. We remark that

$$\xi_1(6) = 0.137 \cdots < \xi(4, 1) = 0.25 < \xi_2(6) = 0.279 \cdots,$$

and that $\omega_2(6, 1) < 4 < \omega_1(6, 1)$ for this case.

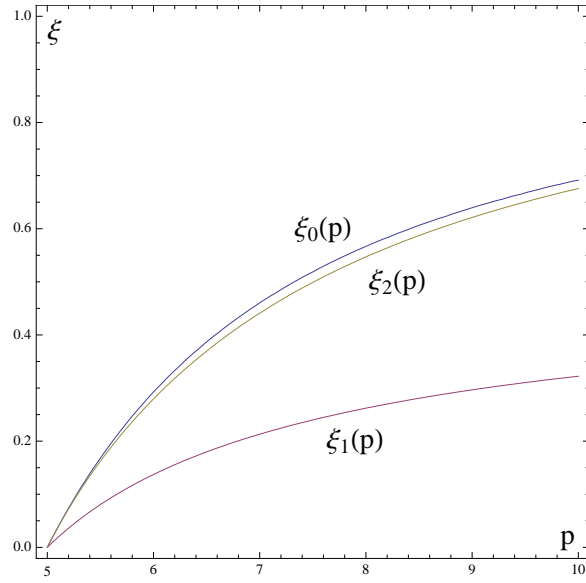


Figure 2. The graphs of $\xi_0(p)$, $\xi_1(p)$ and $\xi_2(p)$ for $5 < p \leq 10$.

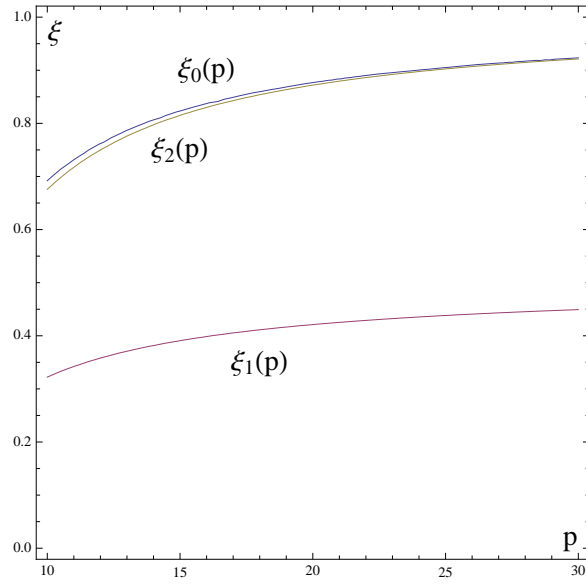


Figure 3. The graphs of $\xi_0(p)$, $\xi_1(p)$ and $\xi_2(p)$ for $10 \leq p \leq 30$.

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